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Embedding 5-cycle systems into pentagon triple systems

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For A.J.W. Hilton in honour of a lifetime of major contributions to finite mathematics

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ABSTRACT

We show that the spectrum for pentagon triple systems is the set of all $n \equiv 1, 15, 21$ or $25 \pmod{30}$. We then construct a 5-cycle system of order $10n+1$ which can be embedded in a pentagon triple system of order $30n+1$ and also construct a 5-cycle system of order $10n+5$ which can be embedded in a pentagon triple system of order $30n+15$, with the possible exception of embedding a 5-cycle system of order 21 in a pentagon triple system of order 61.

1. Introduction

A *Steiner triple system* (or triple system here for short) of order n is a pair (S, T), where T is a collection of edge disjoint triangles (triples) which partitions the edge set of K_n with vertex set S. It is well-known that the spectrum for such triple systems is precisely the set of all $n \equiv 1$ or $3 \pmod 6$ [3]. In this case $|T| = n(n-1)/6$. The three graphs in Fig. 1 are called a *hexagon triple*, an *octagon triple*, and a *pentagon triple* respectively.

The astute reader will notice that each of these graphs is composed of triangles. Therefore if G is any one of the above graphs, an edge disjoint partition of K_n into copies of G is simply an arrangement of the triples of a Steiner triple system into copies of G. This is easier said than done.

So let G be any one of these three graphs. A G-system of order n is a pair $(X, \mathcal{G})$, where $\mathcal{G}$ is a collection of edge disjoint copies of G which partitions the edge set of K_n with vertex set X. If G is a hexagon triple the spectrum for G-systems is the set of all $n \equiv 1$ or $9 \pmod{18}$ [5], and if G is an octagon triple the spectrum for G-systems is the set of all $n \equiv 1$ or $9 \pmod{24}$, $n \geq 25$ [4].

Now if $(X, \mathcal{H})$ is a hexagon triple system of order n, $|\mathcal{H}| = n(n-1)/18$, so the inside triples cannot possibly be a Steiner triple system since $n(n-1)/6$ triples are required. Similarly if $(X, \mathcal{O})$ is an octagon triple system of order n the inside 4-cycles cannot form a 4-cycle system since a 4-cycle system of order n has $n(n-1)/8$ 4-cycles and $|\mathcal{O}| = n(n-1)/24$. So a very natural question to ask is "what is the largest Steiner triple system (4-cycle system) contained in the inside triples (4-cycles) of a hexagon (octagon) triple system?" In [5] it is shown that *every* Steiner triple system of order $n \equiv 1$ or $3 \pmod 6$ can be embedded in the inside triples of some hexagon triple system of order approximately $3n$. In [4] it is shown that there exists a 4-cycle system of every order $n \equiv 1 \pmod 8$ that can be embedded in the inside 4-cycles of some octagon triple system of order approximately $3n$. While not the best possible results (approximately $2n$ is best possible in both cases) they are still quite good.

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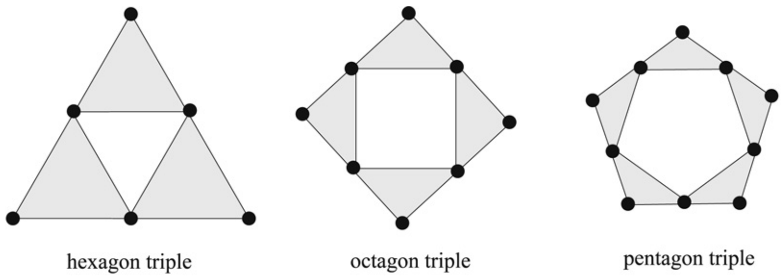

Fig. 1. Hexagon, octagon and pentagon triples.

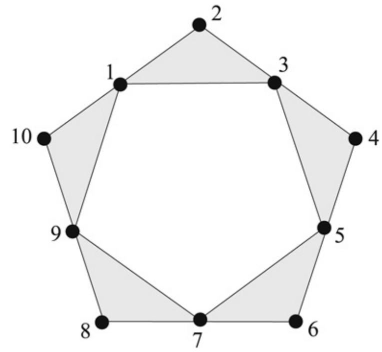

Fig. 2. A labelled pentagon triple.

The object of this paper is to produce a similar result for embedding 5-cycle systems into pentagon triple systems. First we show that the spectrum for pentagon triple systems is precisely the set of all $n \equiv 1, 15, 21$ or $25 \pmod{30}$. We then construct for every positive integer $n \neq 2$ a 5-cycle system of order $10n + 1$ which can be embedded in a pentagon triple system of order $30n + 1$; and for every n we construct a 5-cycle system of order $10n + 5$ which can be embedded in a pentagon triple system of order $30n + 15$. Neither result is best possible. The best possible embedding in each case would be one of approximately twice the size of the 5-cycle system.

2. Important examples

In what follows we will denote the pentagon triple given in Fig. 2 by any cyclic 2-shift of (1, 2, 3, 4, 5, 6, 7, 8, 9, 10) or (1, 10, 9, 8, 7, 6, 5, 4, 3, 2).

The following examples are crucial for all of the results that follow in Sections 3 and 4.

Example 2.1. A pentagon triple system of order 15 containing a 5-cycle system of order 5.
(Z_{15}, P), where

$$\begin{aligned} P = \{&(\mathbf{0}, 12, \mathbf{1}, 5, \mathbf{2}, 6, \mathbf{3}, 10, \mathbf{4}, 11), (\mathbf{0}, 14, \mathbf{3}, 9, \mathbf{1}, 13, \mathbf{4}, 7, \mathbf{2}, 8),\\ &(12, 10, 5, 7, 11, 2, 13, 8, 6, 9),\ (12, 13, 7, 10, 14, 1, 11, 9, 8, 4),\\ &(0, 6, 5, 14, 8, 1, 10, 2, 9, 7),\ (3, 8, 7, 1, 6, 14, 4, 9, 5, 13),\\ &(12, 2, 14, 9, 13, 0, 10, 6, 11, 3)\}.\end{aligned}$$

The inside 5-cycle system of order 5 consists of the two 5-cycles (0, 1, 2, 3, 4) and (0, 3, 1, 4, 2). (See the bold digits above.)

Example 2.2. A pentagon triple system of order 21.
Let the vertex set of K_{21} be $X = \{x_y \mid x \in Z_7,\ y \in \{1, 2, 3\}\}$. Then (X, P) is a pentagon triple system where

$$P = \{(2_3, 2_1, 4_3, 5_1, 3_3, 6_1, 5_2, 1_1, 2_2, 6_3), (0_1, 1_1, 3_1, 1_2, 6_2, 0_3, 5_2, 5_1, 1_3, 2_2)\}.$$

Example 2.3. A pentagon triple system of order 25.
Let the vertex set of K_{25} be $X = \{x_y \mid x \in Z_5,\ y \in \{1, 2, 3, 4, 5\}\}$. Then (X, P) is a pentagon triple system where

$$\begin{aligned} P = \{&(1_3, 3_3, 1_5, 2_5, 2_4, 3_4, 2_2, 0_4, 0_3, 1_2), (0_1, 1_1, 0_2, 2_1, 4_1, 1_2, 2_3, 3_1, 0_4, 1_3),\\ &(0_1, 0_3, 1_4, 2_1, 2_5, 0_2, 1_2, 0_4, 3_4, 1_5), (0_1, 2_3, 3_5, 1_4, 4_3, 1_5, 2_2, 4_2, 2_5, 4_5)\}.\end{aligned}$$

Example 2.4. A pentagon triple system of order 31 containing a 5-cycle system of order 11.
$(\{\infty\} \cup (Z_5 \times \{1, 2, 3, 4, 5, 6\}), P)$, where

$$\begin{aligned} P = \{&(\mathbf{0, 1}), (0, 4), (\mathbf{1, 1}), (1, 4), (\mathbf{2, 1}), (2, 4), (\mathbf{3, 1}), (3, 4), (\mathbf{4, 1}), (4, 4)\} \\ &\cup \{(\infty, (0, 3), (\mathbf{2, 1}), (1, 3), (\mathbf{0, 1}), (1, 5), (\mathbf{2, 2}), (1, 4), (\mathbf{4, 2}), (0, 4)), \\ &((\mathbf{0, 1}), (0, 6), (\mathbf{0, 2}), (1, 5), (\mathbf{1, 2}), (2, 6), (\mathbf{3, 1}), (2, 5), (\mathbf{4, 2}), (1, 6)), \\ &((0, 5), \infty, (0, 6), (2, 6), (0, 4), (2, 4), (3, 3), (1, 6), (3, 5), (1, 3)), \\ &((4, 1), (0, 4), (4, 3), (4, 4), (0, 5), (3, 1), (1, 5), (3, 4), (2, 6), (1, 3)), \\ &((4, 2), (3, 3), (4, 3), (2, 5), (2, 4), (4, 1), (1, 4), (3, 5), (2, 6), (4, 4)), \\ &((0, 2), (1, 3), (3, 3), (1, 2), (0, 6), (0, 3), (4, 6), (2, 1), (2, 5), (3, 4)) \mid \\ &\text{the 1st coordinates are cycled (mod 5) and } \infty \text{ and the 2nd coordinates are fixed}\}. \end{aligned}$$

The inside 5-cycle system of order 11 is $(\infty \cup (Z_5 \times \{1, 2\}), P^*)$, where $P^* = \{(0, 1), (1, 1), (2, 1), (3, 1), (4, 1)\} \cup \{(\infty, (2, 1), (0, 1), (2, 2), (4, 2)),\ ((0, 1), (0, 2), (1, 2), (3, 1), (4, 2)) \mid$ the 1st coordinates are cycled (mod 5) and ∞ and the 2nd coordinates are fixed$\}$. (See the bold entries above.)

Example 2.5. A pentagon triple system of order 51.
Let the vertex set of K_{51} be $X = \{x_y \mid x \in Z_{17},\ y \in \{1, 2, 3\}\}$. Then (X, P) is a pentagon triple system where

$$\begin{aligned} P = \{&(6_3, 16_1, 9_3, 4_1, 13_3, 3_2, 12_1, 0_2, 8_2, 4_2), (0_1, 1_1, 4_1, 9_1, 4_2, 2_1, 10_1, 3_1, 14_3, 6_2), \\ &(0_1, 4_2, 14_2, 2_2, 1_3, 1_1, 0_3, 2_1, 8_3, 3_3), (0_2, 1_2, 1_3, 5_2, 8_3, 3_2, 10_3, 4_3, 15_2, 9_3), \\ &(7_1, 10_2, 16_2, 6_1, 13_2, 12_1, 14_1, 12_2, 9_3, 13_1)\}. \end{aligned}$$

Example 2.6. A pentagon triple system of order 55.
Let the vertex set of K_{55} be $X = \{x_y \mid x \in Z_{11},\ y \in \{1, 2, 3, 4, 5\}\}$. Then (X, P) is a pentagon triple system where

$$\begin{aligned} P = \{&(0_1, 1_1, 3_1, 7_1, 0_2, 2_1, 8_1, 2_2, 3_3, 2_3), (0_1, 0_2, 1_2, 5_1, 7_2, 1_1, 0_3, 2_1, 1_5, 1_3), \\ &(0_2, 2_3, 0_3, 8_5, 4_5, 1_4, 1_3, 4_1, 7_4, 8_2), (0_1, 4_3, 5_4, 5_1, 0_4, 1_1, 6_3, 0_2, 8_4, 3_5), \\ &(0_2, 2_4, 0_5, 1_3, 4_3, 7_4, 9_3, 5_3, 7_5, 3_4), (3_4, 1_1, 10_4, 1_4, 2_3, 3_2, 9_4, 0_5, 2_1, 10_5), \\ &(4_3, 8_2, 9_5, 0_5, 7_1, 7_3, 0_4, 1_5, 7_2, 8_1), (0_2, 0_4, 1_4, 3_2, 0_5, 1_2, 6_4, 3_4, 3_5, 6_5), \\ &(0_1, 0_5, 5_5, 1_2, 3_2, 7_2, 1_3, 9_2, 7_5, 6_5)\}. \end{aligned}$$

Example 2.7. A pentagon triple system of order 61.
(Z_{61}, P), where

$$P = \{(0, 21, 24, 57, 13, 51, 12, 25, 19, 49), (0, 2, 10, 1, 15, 11, 47, 4, 20, 35) \mid \text{ cycled mod } 61\}.$$

Example 2.8. A decomposition of $K_{5,5,5}$ into pentagon triples.
Let $K_{5,5,5,}$ have parts $\{0, 1, 2, 3, 4\}$, $\{5, 6, 7, 8, 9\}$ and $\{10, 11, 12, 13, 14\}$. Then $\{(0, 7, 12, 2, 5, 1, 11, 3, 8, 13), (1, 8, 14, 5, 4, 6, 13, 2, 9, 10), (2, 10, 8, 4, 12, 9, 3, 14, 7, 11), (0, 10, 5, 3, 13, 7, 1, 12, 6, 11), (4, 7, 10, 3, 6, 2, 14, 0, 9, 11)\}$ partitions $K_{5,5,5}$ into pentagon triples.

Example 2.9. A pentagon triple system of order 81.
Let the vertex set of K_{81} be $X = \{x_y \mid x \in Z_{27},\ y \in \{1, 2, 3\}\}$. Then (X, P) is a pentagon triple system where

$$\begin{aligned} P = \{&(0_1, 1_1, 4_1, 9_1, 15_1, 3_1, 2_2, 5_1, 1_2, 8_1), (0_2, 18_3, 17_2, 25_2, 25_1, 24_3, 11_1, 26_2, 19_3, 15_1), \\ &(0_1, 10_1, 3_3, 1_1, 6_2, 20_1, 2_2, 18_1, 1_3, 5_3), (0_2, 4_3, 7_3, 1_2, 16_3, 20_2, 5_3, 11_2, 25_3, 13_3), \\ &(0_1, 6_2, 6_3, 9_1, 17_2, 1_2, 4_2, 10_2, 15_3, 7_3), (15_1, 2_3, 17_1, 0_2, 15_2, 22_2, 17_3, 0_3, 8_1, 6_3), \\ &(0_1, 9_1, 3_2, 1_1, 5_2, 10_1, 0_2, 11_1, 18_2, 0_3), (0_1, 11_3, 16_3, 20_1, 10_3, 0_2, 8_3, 5_2, 22_3, 21_3)\}. \end{aligned}$$

With these examples in hand we can easily determine the spectrum for pentagon triple systems.

3. The spectrum for pentagon triple systems

We deal in turn here with the four congruence classes modulo 30.

3.1. *The* $30k + 1$ *construction*

Since we already have examples for 31 and 61 (Examples 2.4 and 2.7) we can assume $30k + 1 \geq 91$. Let (X, G, B) be a 3-GDD of type 6^k, $k \geq 3$ (see [2]), and set $S = \{\infty\} \cup (X \times \{1, 2, 3, 4, 5\})$. Define a collection of pentagon triples P as follows:

(1) For each group $g \in G$ define a pentagon triple system of order 31 on $\{\infty\} \cup (g \times \{1, 2, 3, 4, 5\})$ (Example 2.4) and place these pentagon triples in P.
(2) For each block $b = \{x, y, z\} \in B$ use Example 2.8 to define a pentagon triple system on $K_{5,5,5}$ with parts $\{x\} \times \{1, 2, 3, 4, 5\}$, $\{y\} \times \{1, 2, 3, 4, 5\}$, and $\{z\} \times \{1, 2, 3, 4, 5\}$, and place these pentagon triples in P.

Then (S, P) is a pentagon triple system of order $30k + 1$.

3.2. *The* $30k + 15$ *construction*

Example 2.1 takes care of order 15, so we can assume $30k + 15 \geq 45$. Let (X, G, B) be a 3-GDD of type 3^{2k+1} (a Kirkman triple system of order $6k + 3$ will do), set $S = X \times \{1, 2, 3, 4, 5\}$, and define a collection of pentagon triples P as follows:

(1) For each group $g \in G$, define a pentagon triple system of order 15 on $g \times \{1, 2, 3, 4, 5\}$ (Example 2.1) and place these pentagons in P.
(2) For each block $b = \{x, y, z\} \in B$, repeat (2) in the $30k + 1$ Construction.

Then (S, P) is a pentagon triple system of order $30k + 15$.

3.3. *The* $30k + 21$ *construction*

Examples 2.2, 2.5 and 2.9 give pentagon triple systems of orders 21, 51 and 81, so we take $k \geqslant 3$.
Let (X, G, B) be a 3-GDD of type $4^1 6^k$ (see [2]) which exists for $k \geqslant 3$, and set $S = \{\infty\} \cup (X \times \{1, 2, 3, 4, 5\})$. Define a collection of pentagon triples P as follows:

(1) Let the one group of size 4 in the GDD be $\overline{g}$. On the set $\{\infty\} \cup (\overline{g} \times \{1, 2, 3, 4, 5\})$, take a pentagon triple system of order 21 (Example 2.2), and place these pentagon triples in P.
(2) For each group g of size 6 in the GDD, on the set $\{\infty\} \cup (g \times \{1, 2, 3, 4, 5\})$, take a pentagon triple system of order 31 (Example 2.4), and place these pentagon triples in P.
(3) For each block $b = \{x, y, z\} \in B$, repeat (2) in the $30k + 1$ Construction.

Then (S, P) is a pentagon triple system of order $30k + 21$.

3.4. *The* $30k + 25$ *construction*

Examples 2.3 and 2.6 give pentagon triple systems of orders 25 and 55, so we take $k \geqslant 2$.
Let (X, G, B) be a 3-GDD of type $5^1 3^{2k}$ (see [2]) which exists for $2k \geqslant 4$, and set $S = X \times \{1, 2, 3, 4, 5\}$. Define a collection of pentagon triples P as follows:

(1) Let the one group of size 5 in the GDD be $\overline{g}$. On the set $\overline{g} \times \{1, 2, 3, 4, 5\}$, take a pentagon triple system of order 25 (Example 2.3), and place these pentagon triples in P.
(2) For each group g of size 3 in the GDD, on the set $g \times \{1, 2, 3, 4, 5\}$, take a pentagon triple system of order 15 (Example 2.1), and place these pentagon triples in P.
(3) For each block $b = \{x, y, z\} \in B$, repeat (2) in the $30k + 1$ Construction.

Then (S, P) is a pentagon triple system of order $30k + 25$.

Theorem 3.1. *The spectrum for pentagon triple systems is precisely the set of all* $n \equiv 1,\ 15,\ 21$ *or* $25 \pmod{30}$.

Proof. This follows from the examples in Section 2 together with the above four constructions. □

All of these constructions will need serious modifications for the embedding results that follow.

4. Two important 3-GDDs

Let $(X, \circ)$ be an idempotent commutative quasigroup of order $2k + 1$, set $S = X \times \{1, 2, 3\}$, and define a 3-GDD (S, G, B) of type 3^{2k+1} as follows:

(1) the groups are $\{(x, 1), (x, 2), (x, 3)\}$ for all $x \in X$, and
(2) if $x \neq y$, the three triples $\{(x, 1), (y, 1), (x \circ y, 2)\}$, $\{(x, 2), (y, 2), (x \circ y, 3)\}$ and $\{(x, 3), (y, 3), (x \circ y, 1)\}$ belong to B.

The astute observer will notice that this is just the Bose Construction in disguise [1]. (See Fig. 3.)
What is important here is that S contains a subset $X \times \{1\}$ with the property that if $(x, 1)$ and $(y, 1)$ belong to $X \times \{1\}$, then the third vertex in the triple containing $(x, 1)$ and $(y, 1)$, namely $\{(x, 1), (y, 1), (x \circ y, 2)\}$, does *not* belong to $X \times \{1\}$. We have the following lemma.

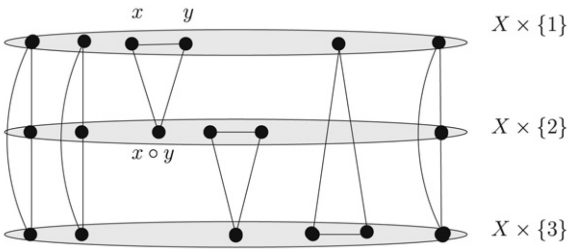

Fig. 3. Illustrating the Bose Construction.

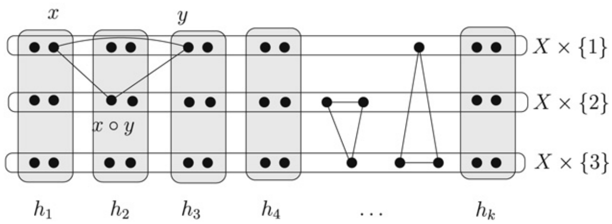

Fig. 4.

Lemma 4.1. *For every odd positive integer* $2k + 1$ *there exist a* 3*-GDD* (S, G, B) *of type* 3^{2k+1} *and a subset* X *of* S *such that* (i) $|X| = 2k + 1$ *and* (ii) *if* $x \neq y \in X$ *and* $\{x, y, a\} \in B$, *then* $a \notin X$. □

Now let $(X, \circ)$ be a commutative quasigroup of order $2k$ with holes $H = \{h_1, h_2, h_3, \ldots, h_k\}$ of size 2, $k \geq 3$. (See [6], for example.) Let $S = X \times \{1, 2, 3\}$ and define a 3-GDD (S, G, B) of type 6^k as follows:

(1) the groups are $h_i \times \{1, 2, 3\}$ for all $h_i \in H$, and
(2) if x and y are in different holes, the three triples $\{(x, 1), (y, 1), (x \circ y, 2)\}$, $\{(x, 2), (y, 2), (x \circ y, 3)\}$ and $\{(x, 3), (y, 3), (x \circ y, 1)\}$ belong to B. (See Fig. 4.)

Note that if x and y belong to different holes, then x, y, and $x \circ y$ are in three different holes. As in the first construction it is important to note that if x and y belong to *different* holes of H, then the third vertex in the triple containing $(x, 1)$ and $(y, 1)$, that is, in the triple $\{(x, 1), (y, 1), (x \circ y, 2)\}$, does not belong to $X \times \{1\}$.

Lemma 4.2. *For every* $2k \geq 6$ *there exist a* 3*-GDD* (S, G, B) *of type* 6^k *and a subset* X *of* S *such that* (i) $|X| = 2k$, (ii) $|X \cap g| = 2$ *for all* $g \in G$, *and* (iii) *if* $x \in X \cap g$ *and* $y \in X \cap h$ *for different groups* g *and* h *and* $\{x, y, a\} \in B$, *then* $a \notin X$. □

With these two lemmas in hand we can proceed to the main results in this paper, that is, the embedding of 5-cycle systems in pentagon triple systems.

5. The 30k + 15 embedding

Write $30k + 15 = 5(6k + 3)$ and let (S, G, B) be a 3-GDD of type 3^{2k+1}, as in Lemma 4.1, with subset $X \subseteq S$, $|X| = 2k + 1$. Let $S^* = S \times \{1, 2, 3, 4, 5\}$ and define a collection of pentagon triples P as follows:

(1) For each $g = \{a, b, x\} \in G$, where $\{x\} = g \cap X$, define a pentagon triple system of order 15 on $g \times \{1, 2, 3, 4, 5\}$ containing the 5-cycle system $\{((x, 1), (x, 2), (x, 3), (x, 4), (x, 5)), ((x, 1), (x, 3), (x, 5), (x, 2), (x, 4))\}$ (Example 2.1) and put these pentagon triples in P.
(2) For each $\{x, y, z\} \in B$, where $|\{x, y, z\} \cap X| \leq 1$ place a copy of Example 2.8 on $K_{5,5,5}$ with parts $\{x\} \times \{1, 2, 3, 4, 5\}$, $\{y\} \times \{1, 2, 3, 4, 5\}$, and $\{z\} \times \{1, 2, 3, 4, 5\}$ and place these pentagon triples in P.
(3) For each $\{x, y, a\} \in B$, where $\{x, y\} \subseteq X$ (and therefore $a \notin X$), use the quasigroup in Fig. 5 to partition $K_{5,5,5}$ with parts $\{x\} \times \{1, 2, 3, 4, 5\}$, $\{y\} \times \{1, 2, 3, 4, 5\}$ and $\{a\} \times \{1, 2, 3, 4, 5\}$ into the 25 triples
 (i) $\{(x, i), (y, j), (a, i \circ j)\}$ (mod 5) or
 (ii) $\{(y, i), (x, j), (a, i \circ j)\}$ (mod 5),
 for all $i, j \in \{1, 2, 3, 4, 5\}$. It is important to note that this construction requires *all* 25 triples to be of type (i) or of type (ii) and *not* some of each. Put another way, we designate one of x and y as "row" and the other as "column".

We will now organize these $25\binom{2k+1}{2}$ triples from (3) into pentagon triples whose inside 5-cycles partition $K^*_{(2k+1)(5)}$ with parts $\{x\} \times \{1, 2, 3, 4, 5\}$, where $x \in X$. (We will denote this graph having $2k + 1$ parts of size 5 by $K^*_{(2k+1)(5)}$.)

To begin we will organize $K^*_{(2k+1)(5)}$ into 5-cycles and then place the triples defined above on the edges of each of these 5-cycles to obtain pentagon triples. Since the number of triples is $25\binom{2k+1}{2}$, the number of edges in $K^*_{(2k+1)(5)}$ is also

$\circ$	1	2	3	4	5
1	1	3	5	2	4
2	5	2	4	1	3
3	4	1	3	5	2
4	3	5	2	4	1
5	2	4	1	3	5

Fig. 5. An idempotent quasigroup of order 5 satisfying $x \circ y \neq y \circ x$.

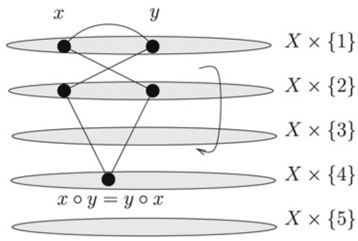

Fig. 6.

$25\binom{2k+1}{2}$, and each triple contains *exactly* one edge of $K^*_{(2k+1)(5)}$, this will partition the type (3) triples into pentagon triples whose inside 5-cycles partition $K^*_{(2k+1)(5)}$. Combining these pentagon triples with the type (1) pentagon triples gives a pentagon triple system of order $30k + 15$ which contains a 5-cycle system of order $10k + 5$ (the pentagon triples of types (1) and (3)).

Now let $(X, \circ)$ be an idempotent commutative quasigroup of order $2k + 1$, and define a collection C of $5\binom{2k+1}{2}$ 5-cycles as follows: for every $x, y \in X$ with $x \neq y$, place the 5-cycles $((x, i), (y, i), (x, i + 1), (x \circ y, i + 3), (y, i + 1)) \pmod 5$ in C.

We will now place triples of type (3) on each of these 5-cycles to form pentagon triples. It suffices to do this for the 5-cycle $((x, 1), (y, 1), (x, 2), (x \circ y, 4), (y, 2))$ (See Fig. 6.). To this end let $\{x, y, a\}$, $\{y, x \circ y, b\}$ and $\{x, x \circ y, c\}$ be the three triples in B containing $\{x, y\}$, $\{y, x \circ y\}$, and $\{x, x \circ y\}$. There are two possibilities for "row" and "column" in each of $\{x, y\}$, $\{y, x \circ y\}$, and $\{x, x \circ y\}$. We will handle the case where x is the row in $\{x, y\}$, y is the row in $\{y, x \circ y\}$, and x is the row in $\{x, x \circ y\}$. The other cases are similar. (Since the type (3) triples form a partial triple system, a, b and c are distinct.) The type (3) triples containing the pairs $\{(x, 1), (y, 1)\}$, $\{(y, 1), (x, 2)\}$, $\{(x, 2)(x \circ y, 4)\}$, $\{(x \circ y, 4), (y, 2)\}$ and $\{(y, 2), (x, 1)\}$ are $\{(x, 1), (y, 1), (a, 1)\}$, $\{(x, 2), (y, 1), (a, 5)\}$, $\{(x, 2), (x \circ y, 4), (c, 1)\}$, $\{(y, 2), (x \circ y, 4), (b, 1)\}$, $\{(x, 1), (y, 2), (a, 3)\}$. Since a, b and c are distinct, placing these five triples on the edges of the 5-cycle $((x, 1), (y, 1), (x, 2), (x \circ y, 4), (y, 2))$ gives a pentagon triple.

Then the pentagon triples (1), (2) and (3) form a pentagon triple system (S^*, P) of order $30k + 15$ and the inside 5-cycles in (1) and (3) form a 5-cycle system of order $10k + 5$. So we have proved the following lemma.

Lemma 5.1. *There exists a* 5*-cycle system of every order* $10k + 5$ *which can be embedded in a pentagon triple system of order* $30k + 15$. $\square$

6. The $30k + 1$ embedding

Write $30k + 1 = 5(6k) + 1$ and let (S, G, B) be the 3-GDD of type 6^k in Lemma 4.2 with $X \subseteq S$ such that (i) $|X| = 2k$, (ii) $|X \cap g| = 2$ for all $g \in G$, and (iii) if $x \in X \cap g$ and $y \in X \cap g'$ for different groups g and g' and $\{x, y, a\} \in B$, then $a \notin X$. This construction is quite similar to the $30k + 15$ Construction and so when appropriate we will use this construction. Let $S^* = \{\infty\} \cup (S \times \{1, 2, 3, 4, 5\})$ and define a collection of pentagon triples P as follows:

(1) For each $g_i \in G$, $i = 1, 2, \ldots, k$, where $h_i = g_i \cap X$, define a pentagon triple system of order 31 on $\{\infty\} \cup (g_i \times \{1, 2, 3, 4, 5\})$ containing a 5-cycle system of order 11 defined on $\{\infty\} \cup (h_i \times \{1, 2, 3, 4, 5\})$ (Example 2.4) and put these pentagon triples in P.
(2) Same as (2) in the $30k + 15$ Construction.
(3) For each $\{x, y, a\} \in B$, where $x \in h_i$ and $y \in h_j$, $i \neq j$, and therefore $a \notin X$, construct 25 triples as in the $30k + 15$ Construction.

We will now organize these $100\binom{k}{2}$ triples in (3) into pentagon triples whose inside 5-cycles partition $K_{k(10)}$ with k parts $h_i \times \{1, 2, 3, 4, 5\}$ of size 10, for $i = 1, 2, \ldots, k$. This construction is exactly the same as in the $30k + 15$ Construction except that we use a commutative quasigroup of order $2k$ with holes of size 2; one exists for all $k \geqslant 3$.

Then the pentagon triples of types (1), (2) and (3) form a pentagon triple system of order $30k+1$, and the inside 5-cycles in (1) and (3) form a 5-cycle system of order $10k+1$.

Lemma 6.1. *There exists a* 5*-cycle system of every order* $10k+1 \neq 21$ *which can be embedded in a pentagon triple system of order* $30k+1$.

Proof. Example 2.4 takes care of order 31. Since there does *not* exist a commutative quasigroup of order 4 with holes of size 2, the $30k+1$ Construction does not work for 61. □

7. Concluding remarks

Combining Lemmas 5.1 and 6.1 gives the following theorem.

Theorem 7.1. *For every* $n \equiv 1$ *or* $5 \pmod{10}$, *except possibly* 21, *there exists a* 5*-cycle system of order* n *which can be embedded in a pentagon triple system of order* $3n-2$ *for* $n \equiv 1 \pmod{10}$ *and* $3n$ *for* $n \equiv 5 \pmod{10}$. □

The following problems remain open.

1. Construct a pentagon triple system of order 61 containing a 5-cycle system of order 21.
2. Improve the above results by embedding a 5-cycle system of order n in a pentagon triple system of order approximately $2n$, rather than the above order of approximately $3n$.
3. Obtain the same results for *any* given 5-cycle system.
4. Construct pentagon triple systems of orders 21 and 25 (mod 30) which contain "large" 5-cycle systems.

References

[1] R.C. Bose, On the construction of balanced incomplete block designs, Ann. Eugenics 9 (1939) 353–399.
[2] Gennian Ge, Group Divisible Designs, in: C.J. Colbourn, J.H. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs, second ed., vol. IV.4, CRC Press, Boca Raton, FL, 2007, pp. 255–260.
[3] T.P. Kirkman, On a problem in combinations, Camb. Dublin. Math. J. 2 (1847) 191–204.
[4] Selda Kücükçifçi, C.C. Lindner, Şule Yazıcı, Embedding 4-cycle systems into octagon triple systems, Ars Combin. (in press).
[5] C.C. Lindner, G. Quattrochi, C.A. Rodger, Embedding Steiner triple systems in hexagon triple systems, Discrete Math. (2008), in press (doi:10.1016/j.disc.2007.12.040) (online).
[6] C.C. Lindner, C.A. Rodger, Design Theory, CRC Press, Boca Raton, FL, 1997.